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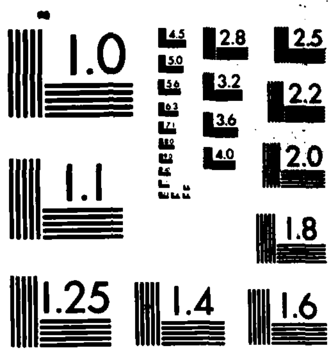

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The research conducted under this effort was centered about general matrix methods and applications of matrix theory in solving large systems of linear equations. In particular, the classification of certain factorizations of M-matrices was undertaken. An extension of the Stein-Rosenberg theorem comparing the spectral radii of matrices useful in constructing iterative solution techniques was obtained. The use of summability methods and approximated conformal mapping techniques in the study of iterative methods was pursued. Finally, a study of SOR and SSOR iterative methods was accomplished using the theory of H-matrices. Eleven papers appeared in the refereed literature during this effort. Keywords:

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Investigations on Improved Iterative Methods for Solving Sparse Systems of Linear Equations

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1. Summary of Research in the Period July, 1981 - June 30, 1984

Broadly speaking, the research supported by the Air Force Office of Scientific Research during this period of nearly three years, has centered about general matrix methods, and applications of matrix theory in solving large systems of linear equations.

Listed below are those research papers, appearing in print in this period (July, 1981 - June, 1984) or pending publication, which were outgrowths of the research supported by the Air Force Office of Scientific Research. (All carry, or will carry, an acknowledgement of AFOSR support.)

1. J. J. Buoni and R. S. Varga, "Theorems of Stein - Rosenberg type. II. Optional paths of relaxation in the complex plane", *Elliptic Problem Solvers* (Martin H. Schultz, ed.), pp. 231 - 240, Academic Press, Inc., New York, 1981.
2. R. S. Varga and D.-Y. Cai, "On the LU factorization of M-matrices", Numer. Math. 38 (1981), 179 - 192.
3. J. J. Buoni, M. Neumann, and R. S. Varga, "Theorems of Stein - Rosenberg type. III. The singular case", Linear Algebra and Appl. 42 (1982), 183 - 198.
4. R. S. Varga and D.-Y. Cai, "On the LU factorization of M-matrices: cardinality of the set $P_n^g(A)$", SIAM J. Algebraic Discrete Methods 3 (1982), 250 - 259.
5. W. Niethammer and R. S. Varga, "The analysis of k-step iterative methods for linear systems from summability theory", Numer. Math. 41 (1983), 177 - 206.
6. W. Gautschi and R. S. Varga, "Error bounds for Gaussian quadrature of analytic functions", SIAM J. Numer. Anal. 20 (1983), 1170 - 1186.
7. R. S. Varga, W. Niethammer, and D.-Y. Cai, "p-cyclic matrices and the symmetric successive overrelaxation method", Linear Algebra and Appl. 58 (1984), 425-439.
8. W. Niethammer, J. de Pillis, and R. S. Varga, "Block iterative methods applied to sparse least squares problems", Linear Algebra and Appl. 58 (1984), 327-341.
9. R. S. Varga, "A survey of recent results on iterative methods for solving large sparse systems of linear equations", Elliptic Problem Solvers II (G. Birkhofff and A. Schoenstadt, eds.), pp. 197-217, Academic Press, Inc., New York, 1984.
10. G. Csordas and R. S. Varga, "Comparisons of regular splittings of matrices", Numer. Math. 44 (1984), 23-35.
11. A. Neumaier and R. S. Varga, "Exact convergence and divergence domains for the symmetric successive overrelaxation (SSOR) iterative method applied to H-matrices", Linear Algebra and Appl. 58 (1984), 261-272.

The above research papers can be roughly grouped into the following areas:

A. Factorization of M-matrics.

B. Extensions of the Stein - Rosenberg Theorem.

C. The use of summability methods and approximate conformal mappings techniques in the study of iterative methods.

D. A study of the SOR (successive overrelaxation) and the SSOR (symmetric successive overrelaxation) iterative methods, using the theory of H-matrices.

E. Comparisons of regular splittings of matrices.

We give below a brief discussion of our research results, according to the topics listed above.

A. Factorization of matrices.

An $n \times n$ M-matrix $A = [a_{i,j}]$ is said to admit an LU factorization into $n \times n$ M-matrices if A can be expressed as

$$A = L \cdot U , \tag{a.1}$$

where $L = [l_{i,j}]$ is an $n \times n$ lower triangular M-matrix (i.e.,

$$l_{i,j} \geq 0 \, , l_{i,j} \leq 0 \text{ for all } i > j \, , \text{ and } l_{i,j} = 0 \text{ for all } j > i ; \; 1 \leq i, \, j \leq n \text{)},$$

and where $U = [u_{i,j}]$ is an $n \times n$ upper triangular M-matrix (i.e.,

$$u_{i,j} \geq 0 \, , \; u_{i,j} \leq 0 \text{ for all } j > i \text{ and } u_{i,j} = 0 \text{ for all } i > j : \; 1 \leq i, \, j \leq n \text{)} .$$

A well-known result from 1962 of Fiedler and Pták [A.1] gives that any **nonsingular** M-matrix admits such an LU factorization in (a.1) with L nonsingular, while in 1977, Kuo [A.3] later showed that any $n \times n$ irreducible M-matrix (singular or not) admits such an LU factorization with, say, L nonsingular, as in (a.1). More recently, in 1981, Funderlic and Plemmons [A.2] have shown that if an $n \times n$ M-matrix A satisfies

$$\underline{x}^T A \geq \underline{0}^T \text{ for some } \underline{x} > \underline{0} , \tag{a.2}$$

then A admits an LU factorization into M-matrices, as in (a.1), with L nonsingular.

What was left open in the literature was an analogous discussion of the LU factorization of reducible and singular M-matrices. This was completely settled by Varga and Cai [A.4], using graph theory. That result is

Theorem 1. ([A.4]). Let A be an $n \times n$ M-matrix. Then, the following are equivalent:

i) **A admits an LU factorization into M-matrices with nonsingular L ;**

ii) **for every proper subset of $\alpha = \{\alpha_1, \alpha_2, \cdots, \alpha_k\}$ of $\{1, 2, \cdots, n\}$ for which the matrix $A[\alpha]$ is singular and irreducible, there is no path in the directed graph $G_n(A)$ of A from vertex v_t to vertex v_{α_j} , for any $t > \alpha_k$ and any $1 \leq j \leq k$.**

It turns out that Theorem 1 also gives the previous results of Fiedler and Pták [A.1] and Kuo [A.3] as special cases.

Next, the condition (a.2) of Funderlic and Plemmons [A.2] implies that

$$\underline{z}^T(PAP^T) \geq \underline{0}^T \text{ , where } \underline{z} := P\underline{x} > \underline{0} , \tag{a.3}$$

for **every** $n \times n$ permutation matrix. In other words, the result of Funderlic and Plemmons [A.2] gives that condition (a.2) is a **sufficient** condition that PAP^T admits an LU factorization into M-matrices with nonsingular L, for **every** $n \times n$ permutation matrix P. It turns out that condition (a.2) is a **necessary** condition for this as well, and this is given Theorem 3 of Varga and Cai [A.4].

Finally, there are easy examples of $n \times n$ M-matrices A which do **not** satisfy (a.3) for every $n \times n$ permutation matrix P. With A an $n \times n$ M-matrix, and with

$$P_n^g(A) := \{n \times n \text{ perm. matrices } P : PAP^T \text{ admits an } LU \text{ factorization with nonsingular } L. \} , \tag{a.4}$$

there then exists $n \times n$ M-matrices A for which $P_n^g(A)$ does **not** contain all $n \times n$ permutation matrices. In Varga and Cai [A.5}, a reduction algorithm (based on the reduced canonical form of the matrix A) is given which either gives the **exact** number of elements in P_n^g, or gives nontrivial upper and lower estimates for the exact number of elements in P_n^g.

References

A.1. M. Fiedler and V. Pták, "On matrices with nonpositive off-diagonal elements and positive principal minors", Czech. Math. J. 12 (1962), 382 - 400.

A.2. R. E. Funderlic and R. J. Plemmons, "*LU* decompositions of M-matrices by elementation without pivoting", Linear Algebra and Appl. 41 (1981), 99 - 110.

A.3. I-wen Kuo, "A note on factorization of singular M-matrices", Linear Algebra and Appl. 16 (1977), 217 - 220.

A.4. R. S. Varga and D.-Y. Cai, "On the *LU* factorization of M-matrices", Numer. Math. 38 (1981), 179 - 192.

A.5. R. S. Varga and D.-Y. Cai, "On the *LU* factorization of M-matrices: cardinality of the set $P_n^g(A)$", SIAM J. Alg. Disc. Meth. 3 (1982), 250 - 259.

B. Extensions of the Stein - Rosenberg Theorem.

To iteratively solve the matrix equation $A\underline{x} = \underline{b}$, where A is a given $n \times n$ complex matrix, it is convenient to express the matrix A as the sum

$$A = D - L - U \tag{b.1}$$

where D, L, and U are $n \times n$ matrices with D assumed nonsingular, and to form the iteration matrices

$$L_\omega := (D - \omega L)^{-1} \left\{ (1-\omega) D + \omega U \right\}, \text{ and } J_\omega := I - \omega D^{-1} A \ . \tag{b.2}$$

Here, L_ω is the familiar successive overrelaxation (SOR) iteration matrix, while J_ω is the extrapolated Jacobi matrix. (The parameter ω is the relaxation factor.) Now, the classical Stein - Rosenberg Theorem [B.4] can be seen to give, **in the case that $J_1 \geq O$ and that $D^{-1}L$ and $D^{-1}U$ are resp. strictly lower and strictly upper triangular matrices,** the following comparison of the spectral radii of these iteration matrices:

$$\begin{cases} \rho(L_\omega) \leq \rho(J_\omega) < 1, \text{ for all } 0 < \omega \leq 1 \text{ if } \rho(J_1) < 1, \\ \rho(L_\omega) \geq \rho(J_\omega) > 1, \text{ for all } 0 < \omega \leq 1 \text{ if } \rho(J_1) > 1 \ . \end{cases} \tag{b.3}$$

Thus, on setting

$$\Omega_L := \left\{ \omega \in \mathbb{C} : \rho(L_\omega) < 1 \right\} ; \mathrm{D}_L := \left\{ \omega \in \mathbb{C} : \rho(L_\omega) > 1 \right\}, \tag{b.4}$$

and

$$\left\{ \Omega_J := \left\{ \omega \in \mathbb{C} : \rho(J_\omega) < 1 \right\} : D_J := \left\{ \omega \in \mathbb{C} : \rho(J_\omega) > 1 \right\}, \quad \text{(b.4}$$

a consequence of the Stein - Rosenberg Theorem can be stated as the

Theorem ([B.4]). Assuming $J_1 \geq O$ and that D is nonsingular in (b.1) with $D^{-1}L$ and DU respectively strictly lower and strictly upper triangular matrices, then

$$\Omega_L \cap \Omega_J \supset (0,1] \ \text{ if } \rho(J_1) < 1 \ , \tag{b.5}$$

and

$$D_L \cap D_L \supset (0,1] \ \text{ if } \rho(J_1) > 1 \ . \tag{b.5$'$}$$

It was this **simultaneous convergence** i.e., $\Omega_L \cap \Omega_J \neq \emptyset$, of the SOR and the extrapolated Jacobi iterative methods in (b.5) which was of interest. One question then is whether this simultaneous convergence of these two iterative methods is valid **without** the assumption that $J_1 \geq O$ and that $D^{-1}L$ and $D^{-1}U$ are triangular. Another question is whether this could be geometrically characterized. These questions were affirmatively answered in Buoni and Varga [B.1].

Theorem ([B.1]). For the splitting of (b.1), assume only that D is nonsingular. Then,

$$\Omega_L \cap \Omega_J \neq \emptyset$$

iff the point $z = 0$ is not contained in $K\ (D^{-1}A)$, the closed convex hull of the eigenvalues of $D^{-1}A$.

Continuing these investigations of generalizations of the Stein - Rosenberg Theorem, suppose that we **have** the simultaneous convergence of the SOR and J_ω iterative methods, i. e. , $O \notin K(D^{-1}A)$ from the previous theorem. This implies that if we consider all relaxation factors ω on the circle $|\omega| = r$, for $r > 0$ sufficiently small, we can attempt to find a unique $\tilde{\Theta}(r)$ such that $\omega = re^{i\tilde{\Theta}(r)}$ **minimizes** the spectral radius $\rho(J_\omega)$ on the circle, and, on joining these points $\{re^{i\tilde{\Theta}(r)}\}$, we can speak of an **optimal path of relaxation**, in the complex plane, for J_ω. One of the main consequences of Buoni and Varga [B.2] is the

Theorem ([B.2]). For the splitting of (b.1), assume that D is nonsingular and that $O \notin K\ (D^{-1}A)$. Then, for each $r > 0$ sufficiently small, there is a unique real $\tilde{\Theta}(r)$ such that

$$\min_{\Theta} \rho\ \left(J_{re^{i\Theta}}\right) = \rho\ \left(J_{re^{i\tilde{\Theta}(r)}}\right) .$$

Moreover, if $re^{i\Psi}$ is the closest point of $K\ (D^{-1}A)$ to the origin, then

$$\lim_{r \to 0} \tilde{\Theta}(r) = -\Psi .$$

Thus, an optimal path of relaxation exists for J_ω, and is tangential to the ray $re^{-i\Psi}$, at the origin. Finally, an optimal path of relaxation for L_ω similarly exists, and is tangential to the optimal path for J_ω , at the origin.

Cases of interest arise where an iteration matrix B is **semiconvergent**, i.e.,

$$\lim_{k \to \infty} B^k \quad \text{exists} . \tag{b.6}$$

In analogy with (b.4), we similarly set (cf. (b.2))

$$S_J := \left\{\omega \in \mathbb{C} : J_\omega \text{ is semiconvergent}\right\} ; \tag{b.7}$$

$$S_L := \left\{\omega \in \mathbb{C} : L_\omega \text{ is semiconvergent}\right\} .$$

For additional notation, let $N\ (B) := \left\{\underline{x} \in \mathbb{C}^n : B\underline{x} = \underline{0}\right\}$ denote the null

space of an $n \times n$ matrix B, and set

$$\text{index}(B) := \min\left\{k : k = 0,1,2,\cdots, \text{ and } N(B^k) = N(B^{k+1})\right\}, \quad \text{(b.8)}$$

where $B^0 := I$. Then, the analogue of the Theorem [B.1] in the singular case is given by Buoni, Neumann, and Varga [B.3]:

Theorem ([B.3]). For the splitting of (b.1), assume that D is nonsingular, and that all eigenvalues of $D^{-1}A$ are not zero. Then,

$$(S_J \cap S_L) \setminus \left\{0\right\} \neq \emptyset$$

iff the point $z = 0$ is not contained in the closed convex hull of the nonzero eigenvalues of $D^{-1}A$, and index $(D^{-1}A) < 1$.

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B.1. J. J. Buoni and R. S. Varga, "Theorems of Stein - Rosenberg type", Numerical Mathematics (R. Ansorge, K. Glashoff, B. Werner, eds.), pp. 65 - 75, Birkhauser Verlag, Basel, 1979.

B.2. J. J. Buoni and R. S. Varga, "Theorems of Stein - Rosenberg type. II. Optimal paths of relaxation in the complex plane", *Elliptic Problem Solvers* (M. H. Schultz, ed.), pp. 231 - 240, Academic Press, Inc., New York, 1981.

B.3. J. J. Buoni, M. Neumann, and R. S. Varga, "Theorems of Stein - Rosenberg type. III. The singular case", Linear Algebra and Appl. 42 (1982), pp. 183 - 198.

B.4. P. Stein and R. Rosenberg, "On the solution of linear simultaneous equations by iteration", J. London Math. Soc. 23 (1948), 111 - 118.

C. The use of summability methods and approximate conformal mapping techniques in the study of iterative methods.

It is well-known in the theory of iterative methods that ideas coming from summability theory can be applied to accelerate standard iterative methods. A good example of this is the concept of **semi-iterative methods** (which was introduced by the author in [C.2]). It is also well-known that the optimization of parameters in an iterative method can often be achieved by means of **conformal mapping techniques**. A good example of this is the optimal determination of the relaxation factor ω for the successive overrelaxation (*SOR*) iteration method, in the two-cyclic consistently ordered case, as first treated in the famous work in 1954 of D. M. Young [C.3]. There, one has the relation

$$(\lambda + \omega - 1)^2 = \lambda\, \omega^2 \mu^2$$

betwen the eigenvalues λ of the *SOR* iteration matrix L_ω and the eigenvalues μ of the associated Jacobi matrix J. The classic result of Young is that the optimum value of ω, ω_b, is given by

$$\omega_b = \frac{2}{1+\sqrt{1-\rho^2(J)}} \tag{c.1}$$

(where $\rho(J)$ denotes the spectral radius of J), when the eigenvalues of J^2 are nonnegative. This result was derived using known conformal properties of the Joukowski mapping.

Deeper theoretical results from summability theory were apparently only recently used in the research of Niethammer and Varga [C.1], where conformal mapping techniques were also brought into play. To briefly describe the results of [C.1], suppose that we are given an $n \times n$ matrix equation $A\,\underline{x} = \underline{b}$, which is reduced to the form

$$\underline{x} = T\,\underline{x} + \underline{c}\ , \tag{c.2}$$

where the $n \times n$ matrix T and the vector $\underline{c}$ are known, and where $\underline{x}$ is the sought vector solution. Now, a k-step stationary iterative method based on (c.2) is

$$\underline{y}_m := \mu_0(T\,\underline{y}_{m-1} + \underline{c}\,) + \mu_1\,\underline{y}_{m-1} + \cdots + \mu_k\,\underline{y}_{m-k}\ ,\ m = k, k+1, \cdots, \tag{c.3}$$

where $\underline{y}_0, \underline{y}_1, \cdots, \underline{y}_{k-1}$ are given starting vectors, and where $\mu_0, \mu_1, \cdots, \mu_k$ are fixed complex numbers (independent of m) which are assumed to satisfy

$$\mu_0 + \mu_1 + \cdots + \mu_k = 1\ . \tag{c.4}$$

The object, of course, is the goal of understanding the theory for selecting the parameters $\{\mu_j\}_{j=0}^{k}$, so as to make the associated error vectors

$$\underline{\tilde{E}}_m := \underline{y}_m - \underline{x}\ ,\ m = k, k+1, \cdots, \tag{c.5}$$

tend to zero as rapidly as possible when $m \to \infty$.

The point of view taken in [C.1] is the following. Given some parameters $\{\mu_k\}_{k=0}^{k}$ satisfying (c.4), what are the **geometrical** conclusions, on the spectrum of eigenvalues of the matrix T of (c.2), so that the error vectors $\{\tilde{\underline{E}}_m\}_{m=k}^{\infty}$ of (c.5) decrease in norm as K^m, as $m \to \infty$, where $0 \leq K < 1$? The technique considered in [C.1} is to use the **general Euler method**, from summability theory, to transform the not necessarily convergent Neumann series $\sum_{j=0}^{\infty} T^j$ for $(I - T)^{-1}$ to a polynomial series $\sum_{j=0}^{\infty} v_j(T)$ with improved convergence properties, where each $v_j(T)$ is a polynomial in the matrix T, of degree at most j. The convergence factor K where K^m measures the norm of $\tilde{\underline{E}}_m$) is then determined by conformal mapping techniques for k-step stationary iterative methods.

The detailed theoretical results of [C.1], along with the necessary notations, are too lengthy to be easily and briefly described here. However, we note that Section 9 of [C.1] is devoted to the study of five examples which connect with well-known results in the theory of iterative methods.

References

C.1. W. Niethammer and R. S. Varga, "The analysis of k-step iterative methods for linear systems from summability theory", Numer. Math. 41 (1983), 177 - 206.

C.2. R. S. Varga, "A comparison of the successive overrelaxation method and semi-iterative methods using Chebyshev polynomials", J. Soc. Indust. Appl. Math. 5 (1957), 34 - 46.

C.3. D. M. Young, "Iterative methods for solving partial differential equations of elliptic type", Trans. Amer. Math. Soc. 76 (1954), 92 - 111.

D. A study of the *SOR* (successive overrelaxation) and the *SSOR* (symmetric successive overrelaxation) iterative methods, using the theory of H-matrices.

To describe our recent research results in this area, we first discuss the results of Varga, Niethammer, and Cai [D.7]. In the iterative solution of the matrix equation

$$A\,\underline{x} = \underline{k}\ , \tag{d.1}$$

assume that the $n \times n$ matrix A is in the partitioned form

$$A = \begin{bmatrix} A_{1,1} & A_{1,2} & 0 & 0 & 0 \\ 0 & A_{2,2} & A_{2,3} & 0 & 0 \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & 0\cdots & A_{p-1,p-1} & A_{p-1,p} \\ A_{1,p} & 0 & 0\cdots & 0 & A_{p,p} \end{bmatrix}, \tag{d.2}$$

where each diagonal submatrix $A_{i,i}$ is square and nonsingular ($1 \le i \le p$, where $p \ge 2$). With $D := \operatorname{diag}\,[A_{1,1};\ A_{2,2};\ \cdots,\ A_{p,p}]$, the associated block Jacobi matrix $B := I - D^{-1}A$ has the (weakly cyclic of index p) form:

$$B = \begin{bmatrix} 0 & B_{1,2} & 0 & \cdots & 0 & 0 \\ 0 & 0 & B_{2,3} & \cdots & 0 & 0 \\ \cdot & & & & \cdot & \cdot \\ \cdot & & & & \cdot & \cdot \\ \cdot & & & & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 0 & B_{p-1,p} \\ B_{p,1} & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}. \tag{d.3}$$

On writing $B = L + U$ (where L and U are respectively strictly lower and strictly upper triangular matrices), the associated *SSOR* iteration matrix S_ω is defined as usual by

$$S_\omega := (I - \omega U)^{-1} \Big[(1-\omega)I + \omega L \Big] (I - \omega L)^{-1} \Big[(1-\omega)I + \omega U \Big] , \tag{d.4}$$

where ω is the relaxation parameter. Then, the new result of [D.7] is the

functional equation

$$\left[\tau - (1-\omega)^2 \right]^p = \tau \left[\tau + 1 - \omega \right]^{p-2} (2-\omega)^2 \omega^p \mu^p , \qquad \text{(d.5)}$$

which couples the eigenvalues τ of S_ω with the eigenvalues μ of B. This functional equation (d.5) has thus the **flavor** of Young's classical result [D.8] of 1954 :

$$(\lambda + \omega - 1)^2 = \lambda \omega^2 \mu^2 ,$$

and its extension by Varga [D.5] in 1959 :

$$(\lambda + \omega - 1)^p = \lambda^{p-1} \omega^p \mu^p ,$$

in the weakly cyclic of index p case, which similarly couples the eigenvalues λ of the *SOR* operator L_ω, defined by

$$L_\omega := (\mathrm{I} - \omega \mathrm{L})^{-1} \left[(1-\omega)\mathrm{I} + \omega \mathrm{U} \right] , \qquad \text{(d.6)}$$

with the eigenvalues μ of the Jacobi matrix B.

To describe an application of the functional equation (d.5) to H-matrices, we first recall some important concepts due to Ostrowski [D.4] in 1937. Any real $n \times n$ matrix $E = [e_{i,j}]$ with $e_{i,j} \leq 0$ for all $i \neq j$, can always be expressed as

$$E = \eta I - C ,$$

where $C = [c_{i,j}]$ is an $n \times n$ matrix having only nonnegative entries. If $\eta > \rho(C)$ (where $\rho(C)$ denotes the spectral radius of C), then E is said to be a **nonsingular** M-matrix. Next, if $F = [f_{i,j}]$ is any $n \times n$ complex matrix, then its associated real $n \times n$ comparison matrix, $M(\mathrm{F}) := [\alpha_{i,j}]$, is defined by

$$\alpha_{i,j} := | f_{i,j} | , \ \alpha_{i,j} = - | f_{i,j} | , \ i \neq j ; 1 \leq i, j \leq n .$$

Then, F is said to be a **nonsingular** H-matrix if its comparison matrix $M(F)$ is a nonsingular M-matrix. For further notation, if $F = [f_{i,j}]$ is any $n \times n$ complex matrix, then $| F |$ denotes the real $n \times n$ matrix $[\, | f_{i,j} | \,]$.

Consider then any $n \times n$ complex matrix $A = [a_{i,j}]$ having nonzero diagonal entries, and set

$$\Omega(A) := \left\{ B = [b_{i,j}] : | b_{i,j} | = | a_{i,j} | \text{ for all } 1 \leq i, j \leq n \right\} . \text{(d.7)}$$

For each $B \in \Omega(A)$, we can express B as

$$B = D(B) - L(B) - U(B) ,$$

where $D(B) := \operatorname{diag} \left[b_{1,1}; b_{2,2}; \cdots , b_{n,n} \right]$ is nonsingular, and where $L(B)$ and $U(B)$ are resectively lower and upper triangular matrices. Then

$$J(B) := \left(D(B) \right)^{-1} \left\{ L(B) + U(B) \right\}$$

defines the associated **Jacobi matrix** for each $B \in \Omega(A)$.

With the notation

$$H_\nu := \left\{ \text{A is an } n \times n \text{ complex matrix}, \; n \text{ arbitrary} : A \text{ is an } H\text{-matrix} \right.$$

$$\left. \text{with } \rho\left(|J(A)| \right) = \nu \right\}, \qquad \text{for each } \nu \in [o, 1),$$

it was shown by Alefeld and Varga in [D.1] in 1976 that, for $A \in H_\nu$, then for any $B \in \Omega(A)$, its associated *SSOR* iteration matrix $S_\omega(B)$ (cf. (d.4)) satisfies

$$\rho\left(S_\omega(B)\right) < 1 \text{ for any } 0 < \omega < \frac{2}{[1+\nu]}, \tag{d.9}$$

i.e., for each B in $\Omega(A)$ and for each ω in $(0, \frac{2}{[1+\nu]})$, $S_\omega(B)$ is **convergent**.

Now, a natural question is if the interval $((0, \frac{2}{1+\nu})$ in ω for convergence in (d.9) is **sharp** for the class H_ν of H-matrices. By applying the functional equation of (d.5) for special matrices in H_ν, it was shown in Varga, Niethammer, and Cai [D.7] that, for each ν with $\frac{1}{2} < \nu < 1$,

$$\sup \left\{ \rho\left(S_\omega(B) \right) : B \in H_\nu \right\} > 1 \quad (\text{ divergence }) \tag{d.10}$$

for any ω satisfying

$$\frac{2}{1+\sqrt{2\nu-1}} < \omega < 1. \tag{d.11}$$

Then, subsequently it was shown in Neumaier and Varga [D.2] (using the theory of regular splittings) that the bound of (d.11) **exactly** separates the convergence and divergence domains for matrices in H_ν. More precisely, if

$$\hat{\omega}(\nu) := \begin{cases} 2, & \text{if } 0 \le \nu \le \frac{1}{2}; \\ \frac{2}{1+\sqrt{2\nu-1}}, & \text{if } \frac{1}{2} < \nu < 1, \end{cases} \tag{d.12}$$

then (cf. [D.2]) for each matrix A in H_ν and for each ω with $0 < \omega < \hat{\omega}(\nu)$,

$$\rho\left(S_\omega(A)\right) < 1. \tag{d.13}$$

Finally, we give brief mention to the fact that similar techniques are applied in Niethammer, de Pillis, and Varga [D.3] to the iterative solution of sparse least

squares problems. A survey of the results of this section can also be found in [D.6].

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E. Comparisons of regular splittings of matrices.

The theory of regular splittings of matrices, introduced in 1960 by Varga [E.2], has been a useful tool in the analysis of iterative methods for solving large systems of equations. For our theoretical background, let A, M, and N be all $n \times n$ complex matrices. Then, $A = M - N$ is a regular splitting of A if M is nonsingular and if M^{-1} and N have all their entries nonnegative, written $M^{-1} \geq O$ and $N \geq O$.

Consider the solution of the matrix problem

$$A \underline{x} = \underline{k} \,, \tag{e.1}$$

where A admits a regular splitting $A = M - N$. Then, (e.1) becomes

$$M \underline{x} = N \underline{x} + \underline{k} \,, \tag{e.2}$$

which induces the iterative method

$$M \underline{x}^{(m+1)} = N \underline{x}^{(m)} + \underline{k} \text{ , or } \underline{x}^{(m+1)} = M^{-1} N \underline{x}^{(m)} + M^{-1} \underline{k} \text{ , } m = 0,1, \tag{e.3}$$

The following are well-known:

Theorem A ([E.2]). Let $A = M - N$ be a regular splitting of A. If $A^{-1} \geq O$, then

$$\rho(M^{-1} N) = \frac{\rho(A^{-1} N)}{1 + \rho(A^{-1} N)} < 1 \,, \tag{e.4}$$

i.e., the iterative method of (e.3) is convergent for any start vector $\underline{x}^{(0)}$. Conversely, if $\rho(M^{-1} N) < 1$, then $A^{-1} \geq O$.

Theorem B ([E.2]). Let $A = M_1 - N_1 = M_2 - N_2$ be two regular splittings of A, where $A^{-1} \geq O$. If $N_2 \geq N_1$ (i.e., $N_2 - N_1 \geq O$), then

$$1 > \rho(M_2^{-1} N_2) \geq \rho(M_1^{-1} N_1) \,, \tag{e.5}$$

i.e., the iterative method (e.3) associated with the splitting $A = M_1 - N_1$ is asymptotically faster than that associated with the splitting $A = M_2 - N_2$.

Less well-known, but nevertheless useful in applications, is the following unpublished thesis result of 1973 of Z. Woźnicki [E.3]:

Theorem C ([E.3]). Let $A = M_1 - N_1 = M_2 - N_2$ be two regular splittings of A, where $A^{-1} \geq O$. If $M_1^{-1} \geq M_2^{-1}$, then

$$1 > \rho(M_2^{-1} N_2) \geq \rho(M_1^{-1} N_1) \,. \tag{e.6}$$

Now, with the hypotheses of Theorem B or C, it is shown in Csordas and Varga [E.1] that

i) $N_2 \geq N_1$ implies $M_1^{-1} \geq M_2^{-1}$;

ii $M_1^{-1} \geq M_2^{-1}$ implies $A^{-1} N_2 A^{-1} \geq A^{-1} N_1 A^{-1}$;

iii) $A^{-1} N_2 A^{-1} \geq A^{-1} N_1 A^{-1}$ implies $(A^{-1} N_2)^j A^{-1} \geq (A^{-1} N_1)^j A^{-1}$ for each positive integer j.

Moreover, the reverse implications in *i*), *ii*), and *iii*) are not in general valid. Now, the weakest hypothesis of the above, namely $(A^{-1} N_2)^j A^{-1} \geq (A^{-1} N_1)^j A^{-1}$ for some positive interger j, gives a generalization of Theorem B and C.

Theorem 1. ([E.1]). Let $A = M_1 - N_1 = M_2 - N_2$ be two regular splittings of A, where $A^{-1} \geq O$. Assume there exists a positive integer j for which $(A^{-1} N_2)^j A^{-1} \geq (A^{-1} N_1)^j A^{-1}$. Then,

$$1 > \rho(M_1^{-1} N_1) \geq \rho(M_2^{-1} N_2). \tag{e.7}$$

A final result of Csordas and Varga [E.1] gives partial converses to Theorems B and C, and unifies the earlier results of Varga and Woźnicki.

Theorem 2 ([E.1]). Let $A = M_1 - N_1 = M_2 - N_2$ be two regular splittings of A, where it is assumed that $A^{-1} > O$. If

$$\rho(M_2^{-1} N_2) > \rho(M_1^{-1} N_1), \tag{e.8}$$

there exists a positive integer j_0 for which (cf. *iii*))

$$(A^{-1} N_2)^j A^{-1} > (A^{-1} N_1)^j A^{-1} \text{ for all } j \geq j_0. \tag{e.9}$$

Conversely, if there is a positive integer j for which

$$(A^{-1} N_2)^j A^{-1} > (A^{-1} N_1)^j A^{-1}, \tag{e.10}$$

then (e.8) is valid.

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